Published by Institute of Physics Publishing for SISSA/ISAS

Received: *January 23, 2003*
Revised: *February 11, 2003*
Accepted: *February 27, 2003*

Exact mesonic vacua from matrix models

Kazutoshi Ohta

Department of Physics, University of Wales Swansea, Swansea, SA2 8PP, UK
E-mail: k.ohta@swansea.ac.uk

Abstract: We investigate in detail the structure of mesonic vacua of $N = 1$ $\mathrm{U}(N_c)$ supersymmetric gauge theory with N_f flavors from the matrix model. We show that the Witten index from the matrix model calculation agrees with a result from field theoretical analysis. We also discuss the relationship between a diagrammatic summation and direct matrix integration with insertion of a variable changing delta function. Using this formalism, we obtain the quantum moduli space and evidence of the Seiberg duality from the matrix models.

Keywords: Supersymmetric Effective Theories, Matrix Models.

Contents

1. Introduction

Dijkgraaf and Vafa recently showed that an effective superpotential of $N = 1$ supersymmetric gauge theory is obtained from a hermitian matrix integral over a tree level superpotential [1, 2, 3]. This observation is originally formulated by using a string theoretical realization of gauge theory, but even apart from string theory the Dijkgraaf-Vafa (DV) procedure itself produces field theoretically a powerful principle to calculate the effective glueball superpotential. After their works, a lot of papers, which discuss from various point of view, have rapidly appeared in [4]–[28].

The model discussed in [1, 2, 3] is mainly the $N = 1$ gauge theory coupled to adjoint chiral superfields with a general superpotential of a polynomial. And also $N = 1^*$ and Leigh-Strassler deformed theory from $N = 4$ theory are discussed in detail by [4, 5, 10]. On the other hand, theory with chiral fields in a fundamental representation, which is the so-called supersymmetric QCD (SQCD) theory, has much interesting dynamics like a generation of the Affleck-Dine-Seiberg (ADS) superpotential [29], Seiberg duality [30, 31, 32], etc. So we expect that the matrix model technique sheds new light on the non-perturbative dynamics of $N = 1$ SQCD and phenomenologically realistic models.

A breakthrough in the calculation of matrix models with flavors has been achieved by [12]. They find that for $\mathrm{U}(N_c)$ gauge theory with N_f flavors in the fundamental representation, the matrix integral is completely performed and infinite series in a perturbative expansion could be represented by a single analytic function. Using their results, a generation of ADS superpotential is discussed in [14, 15]. However, these matrix model calculation has a discrepancy in a sense of the number of supersymmetric vacua (Witten index) compared with results of [33] except for the case of $N_f = 1$ [14]. This implies that we overlook some information in the matrix model calculation.

Moreover, the generation of the ADS superpotential is discussed in a different way [16] by insertion of a matrix valued delta function, which is called as the Wishart integral in the mathematical literature [34, 35]. However, the relationship between these two different methods is still unclear. If we naively use the result from the Wishart integral for the $N_c < N_f$ case, the matrix integral indicates an existence of the Seiberg duality [20, 21]. But we need more careful analysis for this attractive problem.

The purpose of this paper is to resolve the discrepancy of the Witten index between the effective mesonic field theory and matrix model calculation, and to find an exact correspondence between the perturbative series expansion and the direct matrix integration with insertion of the delta function. The former is just done by perceiving ambiguity of the functional representation in a perturbative infinite series. The latter can be found by refining the arguments in [16] from the first principle of the matrix integral. In our argument, we obtain the exact ADS superpotential without the shift of the mass term for the meson field.

The paper is organized as follows. In the subsequent section, we extend the analysis of the effective mesonic vacua which is discussed in [33] for the SU(N_c) case. This analysis tells us a disagreement of the Witten index with the previous matrix model calculation. We reproduce the matrix integration for a general mass matrix in section 3 and we show one to one correspondence between the vacua of the meson and glueball effective theory. In section 4, we discuss in detail the relationship between the results of the perturbative sum of the matrix integration and one from the direct matrix integration. We will give a meaning of the mesonic effective theory in terms of the matrix models.

2. Vacuum structure of effective meson theory

We consider U(N_c) gauge theory with N_f flavors in the fundamental representation in the following. The tree level superpotential of this theory is obtained by a flow from $N = 2$ SQCD theory with fundamental flavors Q_i^a and $\tilde{Q}_i^a$ by adding a mass μ for the adjoint scalar Φ_a^b in $N = 2$ vector multiplet

$$W_{\text{tree}}(\Phi, Q, \tilde{Q}) = \mu \operatorname{Tr} \Phi^2 + \sqrt{2}\tilde{Q}_i^a \Phi_a^b Q_b^i + m_i \tilde{Q}_i^a Q_a^i, \tag{2.1}$$

where $i, j = 1, \ldots, N_f$ stand for the flavor indices and $a, b = 1, \ldots, N_c$ stand for the color indices.

If we integrate out Φ first, we expect the effective superpotential for mesons from the principle of holomorphy and dimensional counting as follows

$$W_{\text{eff}}(M) = -\frac{1}{2\mu} \operatorname{Tr} M^2 + \operatorname{Tr}(mM) + W_{\text{ADS}}\,, \tag{2.2}$$

where we define the gauge singlet meson variable by $M_i^j \equiv Q_i^a \tilde{Q}_a^j$ and W_{ADS} is an instanton induced effective ADS superpotential proposed by [29]

$$W_{\text{ADS}} = (N_c - N_f) \left(\frac{\Lambda_1^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}}, \tag{2.3}$$

with the scale of the $N = 1$ SQCD theory Λ_1. The vacuum structure of this effective theory is carefully analyzed in [33] for SU(N_c) theory. We follow their arguments below for the U(N_c) gauge group.

Using the U(N_f) global symmetry, we can diagonalize M as

$$M = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{N_f})\,. \tag{2.4}$$

Substituting the diagonalized M into the effective superpotential (2.2), we have the vacuum equations for each λ_i

$$-\frac{1}{\mu}\lambda_i + m_i - \frac{\Lambda_1^{(3N_c-N_f)/(N_c-N_f)}}{(\prod_j \lambda_j)^{1/(N_c-N_f)}}\lambda_i^{-1} = 0\,, \tag{2.5}$$

and equivalently this reduces to a quadratic equations

$$\lambda_i^2 - \mu\, m_i \lambda_i + \mu X = 0\,, \tag{2.6}$$

where we define

$$X \equiv \Lambda_1^{(3N_c-N_f)/(N_c-N_f)} \left(\prod_j \lambda_j\right)^{1/(N_f-N_c)}. \tag{2.7}$$

These quadratic equations have two different solutions for each λ_i. We now introduce two indices i and i' which run for solutions with negative and positive signs, respectively, within N_f indices (this choice is for later convenience), that is, we choose

$$\lambda_i = \frac{\mu\, m_i}{2}\left(1 - \sqrt{1-4\alpha_i X}\right), \qquad i \text{ runs for } r \text{ within } N_f \tag{2.8}$$

$$\lambda_{i'} = \frac{\mu\, m_{i'}}{2}\left(1 + \sqrt{1-4\alpha_{i'} X}\right), \qquad i' \text{ runs for } N_f{-}r \text{ within } N_f\,, \tag{2.9}$$

where r is the number of the solutions with negative signature and we define $\alpha_i \equiv 1/\mu m_i^2$.

We now recall that X is defined in (2.7) in terms of λ_i's. So putting (2.8) and (2.9) into the definition of X, we obtain the following equation for each fixed r

$$X = \Lambda_1^{\frac{3N_c-N_f}{N_c-N_f}} \left(\det\left(\frac{\mu\, m}{2}\right)\right)^{\frac{1}{N_f-N_c}} \prod_i \left(1-\sqrt{1-4\alpha_i X}\right)^{\frac{1}{N_f-N_c}} \prod_{i'} \left(1+\sqrt{1-4\alpha_{i'} X}\right)^{\frac{1}{N_f-N_c}}\,, \tag{2.10}$$

and reduce to

$$X^{N_c} = \frac{1}{2^{N_f}}\Lambda^{3N_c} \prod_i \left(1+\sqrt{1-4\alpha_i X}\right) \prod_{i'} \left(1-\sqrt{1-4\alpha_i X}\right), \tag{2.11}$$

where $\Lambda^{3N_c} \equiv (\det\, m)\Lambda_1^{3N_c-N_f}$ is a dynamical scale of the pure $\mathrm{U}(N_c)$ gauge theory.

We consider in the following the case that the mass matrix is proportional to an identity in order to see the explicit structure of solutions in eq. (2.11). In this simple case, the general solution of the eigenvalues preserves a global $\mathrm{U}(r)\times\mathrm{U}(N_f-r)$ symmetry and eq. (2.11) becomes

$$X^{N_c} = \frac{1}{2^{N_f}}\Lambda^{3N_c}\left(1+\sqrt{1-4\alpha X}\right)^r \left(1-\sqrt{1-4\alpha X}\right)^{N_f-r}, \tag{2.12}$$

where all α_i's are identical to α now. If we assume $r > N_f{-}r$ without loss of generality, we can rewrite the vacuum equation into

$$Y^{N_c-r}(1-Y)^{N_c-N_f+r} = \Lambda^{3N_c}\alpha^{N_c}\,, \tag{2.13}$$

where we define $Y \equiv (1+\sqrt{1-4\alpha X})/2$.

We can see that eq. (2.13) is a polynomial of degree $2N_c - N_f$ in Y. So we expect that eq. (2.12) has $2N_c - N_f$ solutions for each fixed r as well as eq. (2.13). The number of ways for choosing r negative signature is given by ${}_{N_f}C_r$, but if we note that the polynomial (2.13) is invariant under exchanging the signs of r positive roots with $N_f - r$ negative ones in eq. (2.12),[1] the essential number of choice is a half of these. So we can count, in terms of X, the total number of vacua (Witten index) as

$$
\begin{aligned}
\mathrm{Tr}(-1)^F &= \frac{1}{2}(2N_c - N_f)\sum_{r=0}^{N_f} {}_{N_f}C_r \\
&= (2N_c - N_f)2^{N_f - 1} \,.
\end{aligned}
\tag{2.14}
$$

In comparison with the results in [12, 15], there exists a discrepancy of the Witten index with the factor of 2^{N_f-1} except for the $N_f = 1$ case [14]. We will refine the matrix model calculation in the next section to resolve this discrepancy.

3. Matrix integral over flavors

We will discuss the derivation of the glueball effective theory from the matrix model procedure given by [1, 2, 3]. We here extend the diagrammatic calculation with flavors in [12] to a general mass matrix.

As conjectured in [12], only the diagram with one flavor index loop contributes to the effective superpotential. A topology of this diagram is a disk with one boundary and we have to sum up all other color index holes. We can insert $S = \mathrm{Tr}\,\lambda^2$ into each color index holes except for one index loop in order to cancel fermionic zero modes. In the diagram which contains only the color index loop, we can choose one hole in which no S operator is inserted among total h holes. Therefore, the contribution to the superpotential is proportional to $N_c h S^{h-1}$, see figure 1, where h in the prefactor is the number of choices of no operator inserted loop and the factor N_c comes from the trace of Chan-Paton color index. This means that the contribution to the effective superpotential from the free energy of the matrix model is $N_c \frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S}$, where $\mathcal{F}_{\chi=2}(S)$ is a sum of the planar diagram with the Euler number $\chi = 2$, or the genus zero.

On the other hand, if we consider the diagrams with one flavor index loop (boundary), we can not insert the operator S into the flavor loop. So we must insert into $h-1$ color indexed holes and there is no other choice of the insertion. This gives just the contribution of $S^{h'}$ to the superpotential, where $h' = h - 1$. So the contribution to the superpotential directly comes from a sum of the diagram with one boundary $\mathcal{F}_{\chi=1}(S)$ instead of the derivative of the free energy. We finally take a trace over the flavor index. This leads to the factor N_f if the system possesses a $\mathrm{U}(N_f)$ global symmetry, but it becomes a sum over the different function with each flavor indices in general, namely $\sum_{i=1}^{N_f} \mathcal{F}^{(i)}_{\chi=1}(S)$, where each function depends on some parameters, which break the global symmetry, like different masses.

[1] There is no symmetry like this for the general mass matrix.

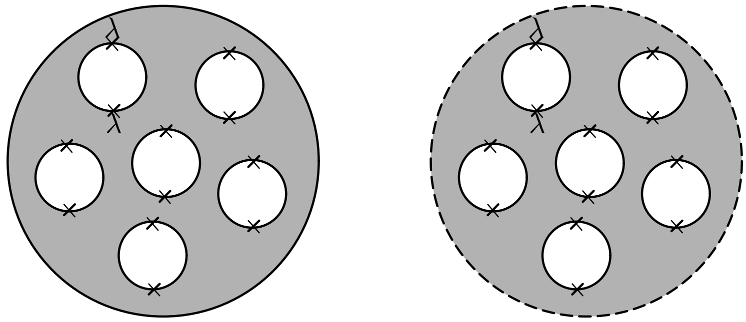

Figure 1: Open string world sheet diagrams of the flavor theory that contribute to the superpotential. We denote color index loops as solid lines and a flavor index loop as a dotted line. There is no operator S insertion in the outer index loop. The contribution from the diagram with one flavor index loop (one boundary) is proportional to $N_f S^{h'}$.

As mentioned above, if we use a mass matrix which is proportional to an identity matrix and preserves an $\mathrm{U}(N_f)$ global symmetry, we have only the contribution from the diagrams with one boundary is always proportional to N_f. This hides the detailed vacuum structure from the matrix model calculation. So we extend the calculation of [12] to the general mass matrix, which is diagonalized by the global symmetry at least, to compare with the field theoretical results in the previous section.

Now let us consider the matrix integral with the tree level superpotential (2.1) to evaluate the contribution from the diagrams with one boundary,

$$Z = \frac{1}{\mathrm{Vol}(\mathrm{U}(N_c))} \int [d\Phi][dQ][d\tilde{Q}] \exp\left\{ -\frac{N_c}{S} W_{\mathrm{tree}}(\Phi, Q, \tilde{Q}) \right\}, \tag{3.1}$$

where $\mathrm{Vol}(\mathrm{U}(N_c))$ is a volume of the gauge group $\mathrm{U}(N_c)$. We regard the quadratic terms of the matrices in $W_{\mathrm{tree}}(\Phi, Q, \tilde{Q})$ as a kinetic term and the Yukawa interaction term as an interaction of the perturbative expansion. We find two point correlators (propagators) from the tree level superpotential as

$$\langle \Phi_a^b \Phi_c^d \rangle = \frac{1}{2\mu} \frac{S}{N_c} \delta_a^d \delta_c^b , \qquad \langle Q_i^a \tilde{Q}_b^j \rangle = \frac{1}{m_i} \frac{S}{N_c} \delta_b^a \delta_i^j . \tag{3.2}$$

The contribution to $\mathcal{F}_{\chi=2}(S)$ comes only from the volume factor and the measure of a gaussian integral of Φ since W_{tree} does not include any self-interacting part of Φ. So the matrix integral (3.1) has the following series expansion with ignoring the volume factor

$$Z = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left(\frac{\sqrt{2} N_c}{S} \right)^{2k} \langle (\tilde{Q}\Phi Q)_1 (\tilde{Q}\Phi Q)_2 \cdots (\tilde{Q}\Phi Q)_{2k} \rangle . \tag{3.3}$$

Using the Feynman rules obtained from the superpotential, we can find weights of the connected diagrams with one boundary and contract each two point correlators, thus we get

$$\frac{N_c}{S} \mathcal{F}_{\chi=1}(S, \mu, m_i) = -\sum_{k=1}^{\infty} \frac{(2k-1)!}{(k+1)!k!} \left(\frac{\sqrt{2} N_c}{S} \right)^{2k} \left(\sum_{i=1}^{N_f} \left(\frac{1}{m_i} \frac{S}{N_c} \right)^{2k} \right) \left(\frac{S}{2\mu N_c} \right)^k N_c^{k+1} . \tag{3.4}$$

Exchanging an order of the summation, we finally obtain

$$\mathcal{F}_{\chi=1}(S,\alpha_i) = -\sum_{i=1}^{N_f}\sum_{k=1}^{\infty}\frac{(2k-1)!}{(k+1)!k!}{\alpha_i}^k S^{k+1}\,, \tag{3.5}$$

where α_i is the same one defined in the previous section.

We now consider the series in (3.5) which is written by a generalized hypergeometric function and has a branch cut discontinuity running from $1/4$ to ∞ in the complex plane. This hypergeometric series could be represented by a standard function as

$$\sum_{k=1}^{\infty}\frac{(2k-1)!}{(k+1)!k!}x^k = \frac{1}{2} - \frac{1}{2f^{(\pm)}(x)} - \log\, f^{(\pm)}(x)\,, \tag{3.6}$$

where $f^{(\pm)}(x)$ are two solutions of the quadratic equation $f^2 - f + x = 0$, namely, defined by

$$f^{(\pm)}(x) \equiv \frac{1\pm\sqrt{1-4x}}{2}\,. \tag{3.7}$$

The sign of $f^{(\pm)}(x)$ depends on which branch x exists.

If we use the above functional representation of the series (3.6), we obtain the free energy from the connected diagrams with one boundary

$$\mathcal{F}_{\chi=1}(S,\alpha_i) = -S\sum_{i=1}^{N_f}\left(\frac{1}{2} - \frac{1}{2f^{(\pm)}(\alpha_i S)} - \log\, f^{(\pm)}(\alpha_i S)\right), \tag{3.8}$$

where the signs are suitably chosen for each i. We notice that the vacuum of [12, 15] represents only the case that all signs are the same, namely the $r=0$ case. This structure of vacua is preserved even though the mass matrix is proportional to the identity matrix.

Therefore finally we obtain the effective superpotential from the above matrix integration in terms of the glueball field

$$\begin{aligned} W_{\text{eff}}(S) &= N_c\frac{\partial\mathcal{F}_{\chi=2}(S)}{\partial S} - 2\pi i\tau_0 S + \mathcal{F}_{\chi=1}(S) \\ &= S\left[N_c\left(1-\log\left(\frac{S}{\Lambda^3}\right)\right) - \sum_{i=1}^{N_f}\left(\frac{1}{2} - \frac{1}{2f^{(\pm)}(\alpha_i S)} - \log\, f^{(\pm)}(\alpha_i S)\right)\right]. \end{aligned} \tag{3.9}$$

The F-flatness equation with respect to S gives

$$S^{N_c} = \frac{1}{2^{N_f}}\Lambda^{3N_c}\prod_i\left(1+\sqrt{1-4\alpha_i S}\right)\prod_{i'}\left(1-\sqrt{1-4\alpha_{i'}S}\right), \tag{3.10}$$

where i and i' run for index of the positive and negative roots, respectively.

This equation is exactly equivalent to eq. (2.11) identifying X with S. So these theories have obviously the same structure of the vacuum solutions. If we especially consider the case that all α_i's are identical, the vacuum equation has $2N_c - N_f$ solutions with each fixed choice of the signs and the number of the independent way to choose the signs is 2^{N_f-1} for the same reason mentioned in the previous section. So we conclude that the Witten index of this theory is

$$\mathrm{Tr}(-1)^F = (2N_c - N_f)2^{N_f-1}\,, \tag{3.11}$$

which agrees with (2.14).

We note here that the above vacuum structure still holds in the limit of $m_i \to 0$ and $\mu \to \infty$ with α_i fixed, namely we can take a massless limit of quarks without the adjoint scalar fields Φ in $N=1$ theory. However if take the limit of $\mu \to \infty$ first, which corresponds to $\alpha_i \to 0$, then the vacuum equation reduces to (no more choice of the signs)

$$S^{N_c} = (\det m)\Lambda_1^{3N_c-N_f}\,, \tag{3.12}$$

which gives only N_c vacua as well as the pure $\mathrm{U}(N_c)$ case. We must be careful that these limits are not commutable.

4. Complete derivation of the ADS superpotential from matrix model

In this section, we will see more explicit correspondence between the effective meson theory and the direct matrix model calculation with a complete proof. We again begin with the matrix integral

$$Z = \frac{\mathcal{N}_Q}{\mathrm{Vol}(\mathrm{U}(N_c))}\int [d\Phi][dQ][d\tilde{Q}]\exp\left\{-\frac{1}{g_s}W_{\mathrm{tree}}(\Phi,Q,\tilde{Q})\right\}, \tag{4.1}$$

where $\mathcal{N}_Q \equiv (\det(m/2\pi g_s))^{N_c}$ is a normalization constant which is determined by the gaussian integral over Q and $\tilde{Q}$.

To compare with the effective theory written by the mesonic variables $M_i^j = Q_i^a\tilde{Q}_a^j$, we consider a change of variables in the superpotential by insertion of a matrix valued delta function

$$e^{-\frac{1}{g_s}W_{\mathrm{tree}}(\Phi,Q,\tilde{Q})} = \int [dM]\delta(M - Q\tilde{Q})e^{-\frac{1}{g_s}W_{\mathrm{tree}}(\tilde{\Phi},M)}\,, \tag{4.2}$$

where

$$W_{\mathrm{tree}}(\tilde{\Phi},M) = \mu\,\mathrm{Tr}\,\tilde{\Phi}^2 - \frac{1}{2\mu}\mathrm{Tr}\,M^2 + \mathrm{Tr}(mM)\,, \tag{4.3}$$

with $\tilde{\Phi}_a^b \equiv \Phi_a^b + \frac{1}{\sqrt{2\mu}}\tilde{Q}_i^b Q_a^i$. The matrix valued delta function is defined by

$$\delta(M - Q\tilde{Q}) = \frac{1}{(2\pi)^{N_f^2}}\int [dT]\exp\left\{i\,\mathrm{Tr}\,T(M - Q\tilde{Q})\right\}, \tag{4.4}$$

where T is an $N_f \times N_f$ matrix. Note that this variable change is valid for $N_c > N_f$ since each matrices have the same degrees of freedom. If $N_c \leq N_f$, then we have to introduce more constraints of variables like baryons.

Substituting (4.2) into (4.1) and integrating gaussian integrals of Q, $\tilde{Q}$ and Φ, we find

$$\begin{aligned} Z &= \frac{\mathcal{N}_Q}{(2\pi)^{N_f^2}\,\mathrm{Vol}(\mathrm{U}(N_c))}\int [d\tilde{\Phi}][dQ][d\tilde{Q}][dM][dT]\exp\left\{-\frac{1}{g_s}W_{\mathrm{tree}}(\tilde{\Phi},M) + i\,\mathrm{Tr}\,T(M - Q\tilde{Q})\right\} \\ &= \frac{\mathcal{N}_Q(2\pi g_s/\mu)^{\frac{1}{2}N_c^2}}{(2\pi)^{N_f^2-N_cN_f}\,\mathrm{Vol}(\mathrm{U}(N_c))}\int [dM][dT]\,(\det(iT))^{-N_c}\exp\left\{-\frac{1}{g_s}W_{\mathrm{tree}}(M) + i\,\mathrm{Tr}\,TM\right\}, \end{aligned} \tag{4.5}$$

where $W_{\mathrm{tree}}(M) = \mathrm{Tr}\left(-\frac{1}{2\mu}M^2 + mM\right)$ is the tree superpotential of the mesonic part only.

Let us consider the following integration over T first. We define an integral

$$I_{N_c,N_f}(M) \equiv \int [dT] \left(\det(iT)\right)^{-N_c} e^{i\,\mathrm{Tr}\,TM} . \tag{4.6}$$

To perform this integral, we decompose the matrices T and M into smaller parts

$$T = \begin{pmatrix} \tau & \boldsymbol{t}^\dagger \\ \boldsymbol{t} & T' \end{pmatrix}, \qquad M = \begin{pmatrix} \rho & \boldsymbol{v}^\dagger \\ \boldsymbol{v} & M' \end{pmatrix}, \tag{4.7}$$

where τ and ρ are scalars, $\boldsymbol{t}$ and $\boldsymbol{v}$ are N_f-1 vectors and T' and M' are $(N_f-1)\times(N_f-1)$ matrices. This decomposition leads to the decomposition of the determinant of iT

$$\det(iT) = \left(i\tau + \boldsymbol{t}^\dagger (iT')^{-1}\boldsymbol{t}\right)\det(iT') . \tag{4.8}$$

Now we can integrate over the real scalar τ by a residue integral

$$\begin{aligned} I_{N_c,N_f}(M) \;=\; & \frac{2\pi}{\Gamma(N_c)}\rho^{N_c-1}\int [dT'] \left(\det(iT')\right)^{-N_c} \exp\left\{i\,\mathrm{Tr}\,T'M'\right\} \times \\ & \times \int d\boldsymbol{t}d\boldsymbol{t}^\dagger \exp\left\{-\rho(\boldsymbol{t}^\dagger+\boldsymbol{a}^\dagger)(iT')^{-1}(\boldsymbol{t}+\boldsymbol{a}) - \frac{1}{\rho}\boldsymbol{v}^\dagger(iT')\boldsymbol{v}\right\}, \end{aligned} \tag{4.9}$$

where $\boldsymbol{a} = -\frac{i}{\rho}(iT')\boldsymbol{v}$. Next if we notice that the integration over $\boldsymbol{t}$ and $\boldsymbol{t}^\dagger$ is a gaussian integral with a constant shift, we obtain the following recursive relation

$$\begin{aligned} I_{N_c,N_f}(M) \;&=\; \frac{(2\pi)^{\frac{1}{2}(N_f+1)}}{\Gamma(N_c)}\rho^{N_c-N_f}\int [dT'] \left(\det(iT')\right)^{N_c-1} \exp\left\{i\,\mathrm{Tr}\,T'\left(M' - \frac{1}{\rho}\boldsymbol{v}\boldsymbol{v}^\dagger\right)\right\} \\ &=\; \frac{(2\pi)^{\frac{1}{2}(N_f+1)}}{\Gamma(N_c)}\rho^{N_c-N_f} I_{N_c-1,N_f-1}\left(M' - \frac{1}{\rho}\boldsymbol{v}\boldsymbol{v}^\dagger\right) . \end{aligned} \tag{4.10}$$

We can easily solve the above recursive relation from the initial condition

$$I_{N_c-N_f+1,1}(\rho) = \frac{2\pi}{\Gamma(N_c-N_f+1)}\rho^{N_c-N_f}$$

if and only if $N_c > N_f$. So we find

$$\begin{aligned} I_{N_c,N_f}(M) \;&=\; \frac{(2\pi)^{\frac{1}{2}N_f^2+\frac{1}{2}N_f}}{\prod_{j=1}^{N_f}\Gamma(N_c-j+1)}\left(\det\, M\right)^{N_c-N_f} \\ &=\; (2\pi)^{\frac{1}{2}N_f^2+\frac{1}{2}N_f}\frac{G_2(N_c-N_f+1)}{G_2(N_c+1)}\left(\det\, M\right)^{N_c-N_f} , \end{aligned} \tag{4.11}$$

where $G_2(z)$ is the Barnes function defined by

$$G_2(z+1) = \Gamma(z)G_2(z) , \qquad G_2(1) = 1 , \tag{4.12}$$

and which has an asymptotic expansion at the large z

$$\log G_2(z+1) \simeq \frac{z^2}{2}\left(\log z - \frac{3}{2}\right) + \frac{z}{2}\log 2\pi - \frac{1}{12}\log z . \tag{4.13}$$

We would like to emphasize here that the formula (4.11) does not contain a term $e^{-\operatorname{Tr} M}$ in contrast with the discussion in [16, 34, 35]. This term has a difficulty in a field theoretical sense since it is proportional to a mass term of the meson field and shifts the meson mass. However, there is no problem in our arguments.

Now if we recall the volume of the $\mathrm{U}(N)$ gauge group is given by [36, 37]

$$\mathrm{Vol}(\mathrm{U}(N)) = \frac{(2\pi)^{\frac{1}{2}N^2+\frac{1}{2}N}}{G_2(N+1)}\,, \tag{4.14}$$

plugging back the formula (4.11) into eq. (4.5), we obtain

$$Z = \frac{\mathcal{N}_Q(2\pi g_s/\mu)^{\frac{1}{2}N_c^2}}{\mathrm{Vol}(\mathrm{U}(N_c{-}N_f))}\int [dM]\,(\det\, M)^{N_c-N_f}\, e^{-\frac{1}{g_s}W_{\mathrm{tree}}(M)}\,. \tag{4.15}$$

The volume factor is changed from $\mathrm{U}(N_c)$ to $\mathrm{U}(N_c - N_f)$. This means that the Veneziano-Yankielowicz term is modified to $(N_c - N_f)S\log S$ due to the matter contribution.

We first consider to complete the above integration over M. In the large N_c, N_f limit with N_f/N_c fixed, the above integration over M can be evaluated by the saddle point approximation. If we again diagonalize M like (2.4),[2] the integral of M is dominated at the solution of the saddle point equations

$$-\frac{1}{\mu}\lambda_i + m_i - S\lambda_i^{-1} = 0\,. \tag{4.16}$$

This quadratic equations has two solutions for each λ_i

$$\lambda_i^{(\pm)} = \mu\, m_i\left(\frac{1\pm\sqrt{1-4\alpha_i S}}{2}\right). \tag{4.17}$$

Picking terms proportional to $1/g_s$ from the approximated M integral part and the volume factor in $-\log Z$, we find exactly

$$\mathcal{F}_{\chi=1}(S,\alpha_i) = -S\sum_{i=1}^{N_f}\left(\frac{1}{2} - \frac{1}{2f^{(\pm)}(\alpha_i S)} - \log\, f^{(\pm)}(\alpha_i S)\right), \tag{4.18}$$

where $f^{(\pm)}(x)$ are defined in (3.7) and signs are properly chosen for each i. This final result, of course, is the same as (3.8) in the previous section since we have just performed the matrix integration over all variables in the alternative way.

We have here integrated all matrix variables to the end, so we have skipped the intermediate mesonic effective theory with the ADS superpotential. To obtain the ADS superpotential from this matrix integral, we have to forget temporally the integral over M in (4.15) and use the DV procedure before the integral of M. We can justify this procedure since the size of M depends only on N_f and the M integral does not contribute to the

[2] This does not bring about the Vandermonde determinant since M is always diagonalized by the fixed $\mathrm{U}(N_f)$ global symmetry.

function of S. We find the expansion of the free energy from $-\log Z$ and get

$$\begin{aligned} W_{\rm eff}(S,M) &= N_c \frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S} - 2\pi i \tau_0 S + \mathcal{F}_{\chi=1}(S) \\ &= S\left[N_c\left(1-\log\left(\frac{S}{\Lambda^3}\right)\right) - N_f(1-\log S) - \log\left(\det(mM)\right)\right] + W_{\rm tree}(M)\,. \end{aligned} \tag{4.19}$$

Next we consider the minimaizing of S, namely if we solve the F-flatness condition with respect to S and substitute into the solution to (4.19), we obtain the effective meson theory

$$W_{\rm eff}(M) = W_{\rm tree}(M) + W_{\rm ADS}(M)\,, \tag{4.20}$$

where $W_{\rm ADS}$ is exactly the same as (2.3) with the identification of the scale $\Lambda_1^{3N_c-N_f} = \Lambda^{3N_c}/\det m$. If we minimize the superpotential (4.20), we apparently see the same vacuum structure as the case where all matrix variables are integrated since we just minimize with respect to M and S in each case.

Now we consider the special case of $N_c = N_f$. We are able to use the above argument for $N_c = N_f$ excluding the baryons, $B = \det Q$ and $\tilde{B} = \det \tilde{Q}$. All logarithmic functions of S disappear from eq. (4.19) in this case. Then we have

$$W_{\rm eff}(S,M) = S\log\left(\frac{\Lambda_1^{2N_c}}{\det M}\right) + W_{\rm tree}\,. \tag{4.21}$$

From the F-flatness equation of S, we obtain the following equation without any gluino condensation

$$\det M = \Lambda_1^{2N_c}\,. \tag{4.22}$$

Therefore we obtain a quantum modification of the classical moduli space from the matrix model [8] when $B = 0$ or $\tilde{B} = 0$.

Finally we note that the integration of the delta function exchanges the volume factor of $\mathrm{U}(N_c)$ for one of $\mathrm{U}(N_c - N_f)$. If we naively apply the formula (4.11) to the case of $N_c < N_f$, we can rewrite eq. (4.15) up to some constants as

$$Z \sim \frac{1}{\mathrm{Vol}(\mathrm{U}(N_f - N_c))}\int [dM][dq][d\tilde{q}]e^{-\frac{1}{g_s}(W_{\rm tree}(M)+\tilde{q}Mq)}, \tag{4.23}$$

where q and $\tilde{q}$ are $(N_f - N_c)\times N_f$ matrices, namely, which represent the fundamental flavors of $\mathrm{U}(N_f - N_c)$ gauge theory. This fact strongly suggests the appearance of the Seiberg duality from the matrix models [20, 21]. Moreover, we comment that the above rough derivation of Seiberg duality is valid even without any mass deformation. However, the degrees of freedom of the quark matrices Q and $\tilde{Q}$ can not be replaced by only the mesonic variables for $N_c \leq N_f$. So we should need more careful analysis with some constraints or baryon matrices [26].

Acknowledgments

I would like to thank N. Dorey, T.J. Hollowood, S.P. Kumar, P. de Madeiros, H. Suzuki and T .Yokono for useful discussions and comments. This work is supported in part by PPARC for Research into Applications of Quantum Theory to Problems in Fundamental Physics.

References

[1] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[2] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[3] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[4] N. Dorey, T.J. Hollowood, S. Prem Kumar and A. Sinkovics, *Exact superpotentials from matrix models*, *J. High Energy Phys.* **11** (2002) 039 [hep-th/0209089].

[5] N. Dorey, T.J. Hollowood, S.P. Kumar and A. Sinkovics, *Massive vacua of $N = 1^*$ theory and S-duality from matrix models*, *J. High Energy Phys.* **11** (2002) 040 [hep-th/0209099].

[6] F. Ferrari, *On exact superpotentials in confining vacua*, *Nucl. Phys.* **B 648** (2003) 161 [hep-th/0210135].

[7] H. Fuji and Y. Ookouchi, *Comments on effective superpotentials via matrix models*, *J. High Energy Phys.* **12** (2002) 067 [hep-th/0210148].

[8] D. Berenstein, *Quantum moduli spaces from matrix models*, *Phys. Lett.* **B 552** (2003) 255 [hep-th/0210183].

[9] R. Dijkgraaf, S. Gukov, V.A. Kazakov and C. Vafa, *Perturbative analysis of gauged matrix models*, hep-th/0210238.

[10] N. Dorey, T.J. Hollowood and S.P. Kumar, *S-duality of the Leigh-Strassler deformation via matrix models*, *J. High Energy Phys.* **12** (2002) 003 [hep-th/0210239].

[11] A. Gorsky, *Konishi anomaly and $N = 1$ effective superpotentials from matrix models*, *Phys. Lett.* **B 554** (2003) 185 [hep-th/0210281].

[12] R. Argurio, V.L. Campos, G. Ferretti and R. Heise, *Exact superpotentials for theories with flavors via a matrix integral*, hep-th/0210291.

[13] J. McGreevy, *Adding flavor to Dijkgraaf-Vafa*, *J. High Energy Phys.* **01** (2003) 047 [hep-th/0211009].

[14] H. Suzuki, *Perturbative derivation of exact superpotential for meson fields from matrix theories with one flavour*, hep-th/0211052.

[15] I. Bena and R. Roiban, *Exact superpotentials in $N = 1$ theories with flavor and their matrix model formulation*, *Phys. Lett.* **B 555** (2003) 117 [hep-th/0211075].

[16] Y. Demasure and R.A. Janik, *Effective matter superpotentials from Wishart random matrices*, *Phys. Lett.* **B 553** (2003) 105 [hep-th/0211082].

[17] S.G. Naculich, H.J. Schnitzer and N. Wyllard, *The $N = 2$ U(N) gauge theory prepotential and periods from a perturbative matrix model calculation*, *Nucl. Phys.* **B 651** (2003) 106 [hep-th/0211123].

[18] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *J. High Energy Phys.* **12** (2002) 071 [hep-th/0211170].

[19] A. Klemm, M. Marino and S. Theisen, *Gravitational corrections in supersymmetric gauge theory and matrix models*, hep-th/0211216.

JHEP02(2003)057

[20] B. Feng, *Seiberg duality in matrix model*, hep-th/0211202.

[21] B. Feng and Y.-H. He, *Seiberg duality in matrix models, II*, hep-th/0211234.

[22] S.G. Naculich, H.J. Schnitzer and N. Wyllard, *Matrix model approach to the $N = 2$ U(N) gauge theory with matter in the fundamental representation*, *J. High Energy Phys.* **01** (2003) 015 [hep-th/0211254].

[23] H. Ita, H. Nieder and Y. Oz, *Perturbative computation of glueball superpotentials for* SO(N) *and* USp(N), *J. High Energy Phys.* **01** (2003) 018 [hep-th/0211261].

[24] H. Itoyama and A. Morozov, *The Dijkgraaf-Vafa prepotential in the context of general Seiberg-Witten theory*, hep-th/0211245.

[25] H. Itoyama and A. Morozov, *Experiments with the WDVV equations for the gluino-condensate prepotential: the cubic (two-cut) case*, *Phys. Lett.* **B 555** (2003) 287 [hep-th/0211259].

[26] I. Bena, R. Roiban and R. Tatar, *Baryons, boundaries and matrix models*, hep-th/0211271.

[27] Y. Ookouchi, *$N = 1$ gauge theory with flavor from fluxes*, hep-th/0211287.

[28] S.K. Ashok, R. Corrado, N. Halmagyi, K.D. Kennaway and C. Romelsberger, *Unoriented strings, loop equations and $N = 1$ superpotentials from matrix models*, hep-th/0211291.

[29] I. Affleck, M. Dine and N. Seiberg, *Dynamical supersymmetry breaking in supersymmetric QCD*, *Nucl. Phys.* **B 241** (1984) 493.

[30] N. Seiberg, *Electric-magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[31] N. Seiberg, *Exact results on the space of vacua of four-dimensional SUSY gauge theories*, *Phys. Rev.* **D 49** (1994) 6857 [hep-th/9402044].

[32] K.A. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric-magnetic duality*, *Nucl. Phys.* **45BC** *(Proc. Suppl.)* (1996) 1 [hep-th/9509066].

[33] G. Carlino, K. Konishi and H. Murayama, *Dynamical symmetry breaking in supersymmetric* SU(N_c) *and* USp($2N_c$) *gauge theories*, *Nucl. Phys.* **B 590** (2000) 37 [hep-th/0005076].

[34] Y.V. Fyodorov, *Negative moments of characteristic polynomials of random matrices: Ingham-Siegel integral as an alternative to Hubbard-Stratonovich transformation*, *Nucl. Phys.* **B 621** (2002) 643 [math-ph/0106006].

[35] R.A. Janik and M.A. Nowak, *Wishart and anti-Wishart random matrices*, math-ph/0112017.

[36] I.G. Macdonald, *The volume of a compact Lie group*, *Invent. Math.* **56** (1980) 93.

[37] H. Ooguri and C. Vafa, *Worldsheet derivation of a large-N duality*, *Nucl. Phys.* **B 641** (2002) 3 [hep-th/0205297].

JHEP02(2003)057